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A non-perturbative approach to critical dynamics

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Abstract
This paper is devoted to a non-perturbative renormalization group (NPRG) analysis of Model A, which stands as a paradigm for the study of critical dynamics. The NPRG formalism has appeared as a valuable theoretical tool to investigate non-equilibrium critical phenomena, yet the simplest—and nontrivial—models for critical dynamics have never been studied using NPRG techniques. In this paper we focus on Model A taking this opportunity to provide a pedagogical introduction to NPRG methods for dynamical problems in statistical physics. The dynamical exponent z is computed in $d = 3$ and $d = 2$ and is found to be in close agreement with results from other methods.

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1. Introduction

The understanding of non-equilibrium critical phenomena stands as one of the major challenges of statistical physics. Systems far from thermal equilibrium are omnipresent in nature: slow relaxation or external driving forces tend to prevent real systems from ever reaching their equilibrium distribution. Behaviour out-of-equilibrium is far richer than at equilibrium, and many intriguing scaling phenomena, such as self-organized criticality (emergence of scaling without fine-tuning of a control parameter) [1], or phase transitions between non-equilibrium stationary states [2–5], have been observed for long. However, despite the considerable achievements of equilibrium statistical physics, the theoretical comprehension of non-equilibrium critical phenomena remains much poorer. The renormalization group (RG), which has appeared as a cornerstone to explain universality in equilibrium continuous phase transitions, has also allowed some breakthroughs out-of-equilibrium [6]. Nonetheless, many non-equilibrium phenomena remain out of range of perturbative approaches because of large coupling constants or because the interesting dimensions lie far from the critical one. Further theoretical progress out-of-equilibrium is hindered by the lack of analytical tools to handle the corresponding models.

Recently, a novel approach—namely the non-perturbative renormalization group (NPRG)—has been proposed to investigate (non-equilibrium) reaction–diffusion processes [7]. It has allowed us to overcome the perturbative limitations and to gain physical insights into models such as branching and annihilating random walks—which are reviewed in [8]. For instance, these studies have captured non-perturbative effects that essentially determine the phase diagram of some systems [7, 9–11] and have unveiled a genuinely non-perturbative fixed point governing the phase transition belonging to the so-called parity-conserving universality class [12]. A valuable advantage of this method is that it gives a unified description of a model: the very same equations enable one to probe any dimensions or coupling regimes, including non-universal features. Hence, the NPRG appears as a powerful tool to investigate non-equilibrium systems.

However, most readers are still largely unfamiliar with these techniques, though they have been introduced more than a decade ago for systems in equilibrium [13, 14]. The aim of this contribution is to give a brief but pedagogical introduction to the NPRG methods for non-equilibrium systems. For this purpose, Model A stands as one of the simplest—yet far from being trivial—dynamical models and it has never been studied within the NPRG framework so far, which we remedy with the present work. We adopt a practical viewpoint and put a particular emphasis on the discussion of the devise of a (non-perturbative) approximation scheme.

The paper is organized as follows. In section 2 we briefly recall the definition of Model A and its critical properties. In section 3, the principles and the construction of the NPRG are outlined for generic non-equilibrium systems and specialized in section 4 to Model A. The corresponding non-perturbative flow equations are derived in section 5 and their numerical integration is dealt with in section 6. The results are eventually discussed in section 7 and followed by a brief summary in section 8.

2. Model A and critical dynamics

The purely dissipative relaxation of a non-conserved field ϕ can be described by the Langevin equation:

$$\partial_t \phi(x,t) = -D\frac{\delta \mathcal{H}[\phi]}{\delta \phi(x,t)} + \eta(x,t) \tag{1}$$

where D denotes a constant and uniform relaxation rate and $\mathcal{H}$ the usual Landau–Ginzburg–Wilson Hamiltonian. On approaching a critical point, the relaxation time of the order parameter starts diverging, which reflects the critical slowing down of the dynamics. The Langevin equation is a 'mesoscopic' description of the system which exploits the associated decoupling of time scales: the order parameter, represented by the (coarse-grained) field ϕ, relaxes much slower than all the other microscopic degrees of freedom, which can hence be modelled by a stochastic Gaussian noise η with zero mean and variance:

$$\langle \eta(x,t)\eta(x',t')\rangle = 2Dk_{\mathrm{B}}T\delta^d(x-x')\delta(t-t'). \tag{2}$$

The strength of the noise is fixed by the Einstein relation which ensures that the system acquires its equilibrium distribution at long time. We here focus on the case of a scalar order parameter with Ising symmetry, described by the Hamiltonian

$$\mathcal{H}[\phi] = \int \mathrm{d}^d x \left(\frac{1}{2}[\nabla\phi(x)]^2 + U(\phi) - h(x)\phi(x)\right), \qquad U(\phi) = \frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4. \tag{3}$$

The Langevin equation (1) corresponds to a Glauber dynamics for the Ising spin and defines Model A in the classification by Halperin and Hohenberg [15]. Besides the equilibrium critical

exponents ν and η, the critical dynamics is characterized by the dynamical exponent z that describes the divergence of the relaxation time $\tau \sim \xi^z \sim |T - T_c|^{-z\nu}$ near the temperature T_c of the second-order phase transition.

2.1. Time reversal symmetry

In the long-time limit after the initial perturbation, the system is expected to become time translational invariant (TTI) and the time-reversal symmetry (TRS) to hold. TRS then yields the fluctuation–dissipation theorem (FDT) which linearly relates the two-point correlation function $C(x - x', t - t') = \langle \phi(x,t)\phi(x',t')\rangle$ with the response function $R(x - x', t - t') = \delta\langle\phi(x,t)\rangle/\delta h(x',t')$ following:

$$R(x - x', t - t') = -\theta(t - t')\frac{1}{T}\partial_t C(x - x', t - t'). \tag{4}$$

In the early stages of the relaxation process, the system is generally not TTI such that R and C may depend on both t and t' and FDT may not hold. The fluctuation–dissipation ratio $TR/\partial_t C$ becomes of particular interest to characterize the violation of FDT and the associated ageing phenomena (see [16] for a recent review and references therein). For our non-perturbative study, we rather focus on the stationary dynamics where the system satisfies TRS.

2.2. Field theory

Any RG treatment starts out from a field theory. Upon introducing a Martin–Siggia–Rose response field $\tilde{\phi}(x,t)$ [17], one can average over the Langevin noise η and cast the stochastic equation (1) into a dynamic functional [18, 19]:

$$\mathcal{S}[\phi,\tilde{\phi}] = \int \mathrm{d}^d x\,\mathrm{d}t\left\{\tilde{\phi}\left[\partial_t\phi + D\frac{\delta\mathcal{H}[\phi]}{\delta\phi}\right] - D\tilde{\phi}^2\right\} \tag{5}$$

(where $k_B T$ has been set to unity). Correlation and response functions can then be expressed as functional averages with the weight $\exp(-\mathcal{S}[\phi,\tilde{\phi}])$.

In this equivalent field theoretical formulation, TRS can be conveniently expressed as an invariance of the action (5) under a specific field transformation, as stressed in [20]. This transformation writes

$$\begin{cases} t \to -t \\ \phi \to \phi \\ \tilde{\phi} \to \tilde{\phi} - \dfrac{1}{D}\partial_t\phi, \end{cases} \tag{6}$$

that is one has $\mathcal{S}_t[\phi,\tilde{\phi}] = \mathcal{S}_{-t}\big[\phi, \tilde{\phi} - \frac{1}{D}\partial_t\phi\big]$. Indeed, one can straightforwardly check that after performing a time inversion $t \to -t$ in (5) which switches the sign of the kinetic term $\tilde{\phi}\partial_t\phi$, the field transformation (6) yields additional contributions from the latter term and from the noise term $D\tilde{\phi}^2$ that cancel out. Besides, the transformation of the Hamiltonian part under (6) produces an additional term $\propto \partial_t\phi\delta\mathcal{H}/\delta\phi$, which vanishes upon time integration in the stationary regime. We shall rely in the following on this simple expression of TRS to ensure that this invariance is preserved within the non-perturbative formulation.

3. The NPRG formalism in non-equilibrium statistical physics

The NPRG formalism relies on Wilson's RG idea [21], which consists in building a sequence of scale-dependent effective Hamiltonians, that interpolate smoothly between the short-distance

physics at the microscopic (momentum) scale $k = \Lambda$ and the long-distance physics at the scale $k = 0$, through progressively averaging over fluctuations. Rather than expressing—as in the original Wilsonian formulation—the flow of effective Hamiltonians for the slow modes, one can work out the flow of effective 'Gibbs free energies' Γ_k for the rapid ones, following [22, 23]. Γ_k thus only includes fluctuation modes with momenta $|q| \geqslant k$. At the scale $k = \Lambda$, no fluctuation is yet taken into account and Γ_Λ coincides with the microscopic action $\mathcal{S}$ [22], while at $k = 0$, all fluctuations are integrated out and Γ_0 is the analogue of the Gibbs free energy Γ at thermal equilibrium, in that it encompasses the long-distance and long-time properties of the system. Hence, to construct Γ_k, one needs at a given scale k to suppress the slow modes with momentum $|q| < k$. This is achieved by adding to the original action (5) a scale-dependent term [7, 23, 24] which is quadratic in the fields (so as to affect the propagator of the corresponding modes):

$$\Delta\mathcal{S}_k[\phi, \tilde{\phi}] = \frac{1}{2}\int_{x,t} [\phi(x,t), \tilde{\phi}(x,t)]\hat{R}_k(\nabla^2, \partial_t)^t[\phi(x,t), \tilde{\phi}(x,t)], \tag{7}$$

where $\hat{R}_k$ is a symmetric 2×2 matrix of elements R_k^{ij} $(i, j = 1, 2)$. These elements (so-called cutoff functions) will be specified in the following, but their general properties can be already stressed. In order to achieve the renormalization procedure outlined above, these cutoff functions must behave at fixed k as $R_k^{ij} \sim k^2$ (in Fourier space) for small momenta $|q| \leqslant k$—so that the slow fluctuation modes acquire a 'mass' k^2 and decouple. On the other hand, R_k^{ij} must vanish for large momenta $|q| \geqslant k$—so that the rapid modes remain unaltered and contribute to the functional averages. Besides, the additional constraints

$$\lim_{k\to 0} R_k^{ij} = 0, \qquad \lim_{k\to\Lambda} R_k^{ij} = \infty \quad \text{at fixed } q \tag{8}$$

must be satisfied in order to enforce the correct asymptotic behaviours at the scales $k = \Lambda$ and $k = 0$, respectively $\Gamma_{k=\Lambda} \sim \mathcal{S}$ and $\Gamma_{k=0} = \Gamma$ [7, 23, 24].

With the additional term (7) the 'partition functions'

$$\mathcal{Z}_k[j, \tilde{j}] = \int \mathcal{D}\phi\mathcal{D}i\,\tilde{\phi}\exp\left(-\mathcal{S} - \Delta\mathcal{S}_k + \int j\phi + \int \tilde{j}\tilde{\phi}\right) \tag{9}$$

become k-dependent. Finally, the effective Γ_k which is the central object of the NPRG procedure is defined as the (modified) Legendre transform of $\log \mathcal{Z}_k[j, \tilde{j}]$:

$$\Gamma_k[\psi, \tilde{\psi}] + \log \mathcal{Z}_k[j, \tilde{j}] = \int j\psi + \int \tilde{j}\tilde{\psi} - \Delta\mathcal{S}_k[\psi, \tilde{\psi}]. \tag{10}$$

Γ_k is a functional of the conjugate fields $\psi = \delta \log \mathcal{Z}_k/\delta j$ and $\tilde{\psi} = \delta \log \mathcal{Z}_k/\delta \tilde{j}$. The additional term $\Delta\mathcal{S}_k$ in equation (10) is necessary to set the proper microscopic behaviour at $k = \Lambda$: $\Gamma_{k=\Lambda} \sim \mathcal{S}$ [23]. The RG flow of Γ_k under an infinitesimal change of the scale k—or rather $s = \log(k/\Lambda)$—is governed by an exact functional differential equation [23, 24] (which is derived in the appendix):

$$\partial_s\Gamma_k = \frac{1}{2}\mathrm{Tr}\int_{q,\omega} \partial_s\hat{R}_k\left(\hat{\Gamma}_k^{(2)} + \hat{R}_k\right)^{-1}. \tag{11}$$

In this equation, $\hat{\Gamma}_k^{(2)}[\psi, \tilde{\psi}]$ is the 2×2 matrix of second derivatives of Γ_k with respect to (wrt) ψ and $\tilde{\psi}$ and $\left[\hat{\Gamma}_k^{(2)} + \hat{R}_k\right]^{-1}$ hence embodies the full (functional) propagator associated with the effective theory $\mathcal{S} + \Delta\mathcal{S}_k$.

Obviously, equation (11) cannot be solved exactly and one has to resort to some approximations [23]. However, as the approximations used do not rely on the smallness of a parameter (see the next section), the approach remains non-perturbative in essence. In particular, it is not confined to weak-coupling regimes or to the vicinity of critical dimensions and is therefore suitable to overcome the limitations of perturbative RG schemes.

4. NPRG for Model A

To exploit the exact flow equation (11), one has to devise an approximation scheme. This scheme is based on the construction of an *Ansatz* for Γ_k which does not spoil the non-perturbative features of the exact equation—and which can be systematically enlarged. The formulation of this Ansatz relies on the physics one wishes to probe, that is some basic physical insights are necessary. The most common truncation consists in expanding Γ_k in powers of gradients [22] and time derivatives. The accuracy and convergence of this approximation scheme have been thoroughly studied in the equilibrium context and have shown that quantitatively reliable results can already be obtained at the leading order (∇^2) [23]. For instance, for the three-dimensional Ising model, NPRG calculations yield for the critical exponents $\nu = 0.628$ and $\eta = 0.0443$ at order ∇^2 [23, 25], and $\nu = 0.632$ and $\eta = 0.033$ at order ∇^4 [25] which are in close agreement with the 6-loop results $\nu = 0.6304(13)$ and $\eta = 0.0335(25)$ [26]. Another useful approximation scheme is the field expansion of Γ_k. This truncation has the advantage of preserving the momentum structure of higher order vertices but it approximates the functional structure of the effective potential [23]. The derivative expansion is best appropriate for the study of critical physics which is conveyed by the large-distance ($q \to 0$) and long-time ($\omega \to 0$) modes. We hence adopt here this approximation scheme and expand Γ_k at leading order in derivatives—i.e. only terms of order ∇^2 and ∂_t are retained.

4.1. Construction of an Ansatz for Γ_k at leading order

The form of the Ansatz for Γ_k appropriate to Model A is further dictated by the symmetries. Since we consider the long-time regime where TRS holds, we want the Ansatz to be invariant under the field transformation (6) (where D is set to unity), which in turn imposes the following structure:

$$\Gamma_k(\psi, \tilde{\psi}) = \int \mathrm{d}^d x \, \mathrm{d}t \left\{ \tilde{\psi} X_k^a(\psi) \partial_t \psi + \tilde{\psi} X_k^b(\psi, \nabla\psi) - X_k^c(\psi, \nabla\psi) \tilde{\psi}^2 \right\}. \quad (12)$$

No higher powers of the response field are allowed at this order due to TRS. Indeed, the transformation (6) would connect a generic 'noise' term $\tilde{\psi}^n X_k^d$, $n > 2$ to higher order kinetic terms $\tilde{\psi}^j (\partial_t \psi)^{n-j} X_k^d$, $j = 0, \ldots, n$ which are discarded at order ∂_t. Further constraints on the X_k^i's, $i = a, b, c$ can be deduced from TRS in the same way as in section 2. First one must have $X_k^c = X_k^a \equiv X_k$ for the additional contributions generated by the transformation (6) of the 'noise' and the 'kinetic' terms to cancel out. As for the remaining linear term in $\tilde{\psi}$, X_k^b should write as a (field) derivative of a functional $X_k^b(\psi, \nabla\psi) \equiv \delta \mathcal{F}_k / \delta \psi$ for its transform under (6) to vanish upon time integration. We naturally adopt for $\mathcal{F}_k$ the usual equilibrium Ansatz at order ∇^2 for the Ising model which has been widely studied in the past [23, 25]:

$$\mathcal{F}_k[\psi] = \int \mathrm{d}^d x \left\{ \frac{1}{2} Z_k(\psi) [\nabla\psi]^2 + U_k(\psi) \right\}. \quad (13)$$

In this Ansatz, the functional U_k embodies the effective potential and the renormalization function Z_k encompasses the anomalous dimension of the field (see below).

Furthermore, at the scale $k = \Lambda$, Γ_Λ must identify with the microscopic (bare) action (5)—up to the response field rescaling $\tilde{\psi} \to \tilde{\psi} / X_\Lambda$—i.e. one has

$$X_\Lambda \equiv \frac{1}{D}, \qquad Z_\Lambda \equiv 1, \qquad U_\Lambda \equiv U(\phi) \qquad (U(\phi) \text{ is defined in equation (3)}). \quad (14)$$

The Ansatz for Model A at leading order finally writes

$$\Gamma_k(\psi,\tilde{\psi}) = \int_{x,t} \tilde{\psi} X_k \partial_t \psi - \tilde{\psi}\left[Z_k(\psi)\nabla^2\psi + \frac{1}{2}\partial_\psi Z_k(\psi)[\nabla\psi]^2 \right] + \tilde{\psi}\partial_\psi U_k(\psi) - X_k\tilde{\psi}^2. \quad (15)$$

This Ansatz constitutes the basis of our work.

4.2. Definition of the critical exponents

We now discuss how the critical exponents can be computed within the NPRG approach. In the critical regime, Z_k is expected to endow a scaling form $\bar{Z}_k \sim k^{-\eta_Z}$ and following ψ is expected to scale as $\psi \sim k^{(d-2+\eta_Z)/2}$. The critical exponent η hence corresponds to $\eta_Z = -\partial_s \ln \bar{Z}_k$ at the critical point [23]. Similarly, X_k is expected to scale as $\bar{X}_k \sim k^{-\eta_X}$ at the critical point such that $\omega \sim k^{2-\eta_z+\eta_X}$ according to a scaling analysis of equation (15). Hence, the dynamical exponent z, which by definition characterizes the divergence of the time scale following $\omega = k^z$, is given by $z = 2 - \eta_Z + \eta_X$ where $\eta_X = -\partial_s \ln \bar{X}_k$.

Note that $Z_k(\psi)$ is a function of ψ whereas the scaling form $\bar{Z}_k$ should be a mere (k-dependent) number. In general, one defines $\bar{Z}_k$ as the value of $Z_k(\psi)$ at a given point ψ_0, $\bar{Z}_k \equiv Z_k(\psi_0)$. Similarly, though X_k here is not a function, its flow equation depends on $\psi, U_k(\psi), Z_k(\psi)$ and their derivatives. Henceforth, the notation $\bar{X}_k$ will mean that the corresponding expressions are evaluated for $\psi = \psi_0$. Of course, within the *exact* renormalization flow, the critical exponents should not *in fine* depend on the choice of ψ_0. However any approximation introduces a residual dependence and the choice of ψ_0 may become important. The advocated choice (from equilibrium studies) is the (running) minimum of the effective potential U_k which is implicitly defined by $\partial_\psi U_k(\psi_0) = 0$, for it possesses the best 'stability' properties [23].

4.3. Cutoff matrix

Our last discussion to complete the settings of the NPRG formalism for Model A concerns the choice of the cutoff matrix $\hat{R}_k$. The previous symmetry requirements obviously also apply for the quadratic term $\Delta\mathcal{S}_k$, which must in particular be invariant under TRS (in the stationary regime).

The minimal non-perturbative renormalization scheme consists in performing a space coarse-graining on the propagator mode $\tilde{\psi}\psi$, which amounts to considering an off-diagonal cutoff matrix $\hat{R}_k$ with elements

$$R_k^{12} = R_k^{21} \equiv Q_k(q^2) \qquad \text{and} \qquad R_k^{11} = R_k^{22} = 0. \quad (16)$$

This form for $\hat{R}_k$ is the most natural extension of the equilibrium case—where one considers the effective theory $\mathcal{H}+\Delta\mathcal{H}_k$ where $\Delta\mathcal{H}_k = 1/2\int_q \phi(-q)Q_k(q^2)\phi(q)$ is the scale-dependent quadratic term introduced in the (equilibrium) partition function to achieve the splitting of the fluctuation modes [23]. The choice (16) simply corresponds to $\Delta\mathcal{S}_k \propto \delta\Delta\mathcal{H}_k/\delta\phi$. We recall that the cutoff function $Q_k(q^2)$ must decay fast for large momentum modes and behave as k^2 for slow modes as emphasized in section 3. A typical cutoff function which has been widely used since it allows for analytical results is the θ cutoff introduced by Litim [27]:

$$Q_k(q^2) \propto (k^2 - q^2)\theta(k^2 - q^2) \quad (17)$$

where θ is the Heaviside step fucntion.

It turns out that, even when considering the dynamics, space coarse-graining is enough to achieve a proper non-perturbative renormalization program since the frequency integrals appear to be convergent and need not be regularized [7]. The cutoff matrix (16) has hence been

adopted in all previous non-equilibrium studies and more specifically for reaction–diffusion processes [8].

Note that, on the other hand, one could expect that a time coarse-graining on the $\psi\tilde{\psi}$ propagator may improve the procedure, though it has never been tested. A time coarse-graining could be achieved by adding a frequency cutoff $R_k^{12} = \Omega_k(\mathrm{i}\omega)$ on $\psi\tilde{\psi}$ modes. But in this case one would have to coarse-grain the noise part correspondingly in order to sustain TRS (i.e. the invariance under the transformation (6) of $\Delta\mathcal{S}_k$). This would amount to introducing an additional cutoff on $\tilde{\psi}\tilde{\psi}$ modes, of the form $R_k^{22} = -2\mathrm{i}/\omega\Omega_k(\mathrm{i}\omega)$. The properties of this mixed regularization scheme have never been investigated as yet and is left for future work since it represents a great deal of numerical efforts.

5. Flow equations

The NPRG flow equations for the renormalization functions U_k, Z_k and X_k are drawn from the exact flow of Γ_k given by equation (11). According to the Ansatz (15), $\partial_\psi U_k$ can be defined by

$$\partial_\psi U_k = \frac{1}{(2\pi)^{d+1}\delta^{d+1}(0)} \lim_{p,\nu\to 0} \frac{\delta\Gamma_k}{\delta\tilde{\psi}(p,\nu)}\bigg|_{\tilde{\psi}=0} \tag{18}$$

where the limit of vanishing external momentum and frequency $p, \nu \to 0$ means that the fields are evaluated in uniform and stationary configurations and the prefactor just corresponds to the volume of the system in Fourier space. Similarly, the renormalization functions Z_k and X_k can be defined by

$$Z_k = \frac{(2\pi)^{d+1}}{\delta^{d+1}(0)} \lim_{p,\nu\to 0} \partial_{p^2} \frac{\delta^2\Gamma_k}{\delta\tilde{\psi}(p,\nu)\delta\psi(-p,-\nu)} \tag{19}$$

$$X_k = \frac{(2\pi)^{d+1}}{\delta^{d+1}(0)} \lim_{p,\nu\to 0} \partial_{\mathrm{i}\nu} \frac{\delta^2\Gamma_k}{\delta\tilde{\psi}(p,\nu)\delta\psi(-p,-\nu)}. \tag{20}$$

Obviously, X_k can alternatively be defined from the noise part as $X_k \propto \delta^2\Gamma_k/\delta\tilde{\psi}^2$ for a uniform and stationary configuration. One can check that both definitions lead to the same flow equation $\partial_s X_k$ which in turn reflects that TRS is preserved by the NPRG flow at any scale s.

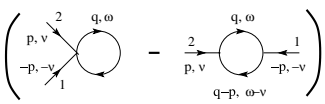
The flow equations of the renormalization functions $\partial_\psi U_k$, Z_k and X_k are obtained by taking the scale derivative ∂_s of the expressions (18), (19) and (20), respectively. It is convenient to first rewrite the flow equation $\partial_s\Gamma_k$ (given by (11)) as

$$\partial_s\Gamma_k = \tfrac{1}{2}\tilde{\partial}_s \mathrm{Tr}\ln\left(\hat{\Gamma}_k^{(2)} + \hat{R}_k\right), \tag{21}$$

where $\tilde{\partial}_s(.) \equiv \hat{R}_k\delta(.)/\delta\hat{R}_k$ only acts on the s-dependence of the cutoff elements R_k^{ij}. It follows that the field derivatives of $\partial_s\Gamma_k$ admit simple diagrammatic representations:

$$\frac{\delta\partial_s\Gamma_k}{\delta\tilde{\psi}(0,0)} = \frac{1}{2}\tilde{\partial}_s$$

$$\frac{\delta^2\partial_s\Gamma_k}{\delta\tilde{\psi}(p,\nu)\delta\psi(-p,-\nu)} = \frac{1}{2}\tilde{\partial}_s$$



In these graphs, the index 1 or 2 on external legs refers to the corresponding field (ψ or $\tilde{\psi}$ respectively), the n-point vertices correspond to 2×2 matrices of $\Gamma_k^{(n)}$ with $n-2$ (external)

fixed indices and two summed over. The propagator lines stand for $[\hat{\Gamma}_k^{(2)} + \hat{R}_k]^{-1}$, which can easily be computed from (15) and (16). We do not detail the full computation of these graphs nor the subsequent explicit frequency integration which are lengthy but straightforward.

Before giving the resulting flow equations, let us put them in a suitable form for the search of fixed point—since we are interested *in fine* in the scale invariant regime. First it is convenient to explicitly express the Ising symmetry by defining the functions $\tilde{U}_k(\rho) \equiv U_k(\psi)$, $\tilde{Z}_k(\rho) \equiv Z_k(\psi)$ where $\rho = \psi^2/2$ is the Z_2 invariant. The derivatives of the functions $\tilde{U}_k(\rho)$ and $U_k(\psi)$ are simply related: $\partial_\psi U_k = \psi\,\partial_\rho U_k$, $\partial_\psi^2 U_k = \partial_\rho U_k + 2\rho\partial_\rho^2 U_k \cdots$ and similarly for Z_k and $\tilde{Z}_k$. As for the flow equation, $\partial_s(\partial_\rho \tilde{U}_k) = 1/\psi\,\partial_s(\partial_\psi U_k)$ for nonzero ψ and $\partial_s \tilde{Z}_k = \partial_s Z_k$. Then to absorb any explicit dependence on the running scale k, we introduce the dimensionless quantities (according to (15))

$$\begin{cases} \bar{\rho} = k^{2-d}\bar{Z}_k \rho \\ u(\bar{\rho}) = k^{-d}\tilde{U}_k(\rho) \\ z(\bar{\rho}) = \bar{Z}_k^{-1}\tilde{Z}_k(\rho) \end{cases} \tag{22}$$

where $\bar{Z}_k^{-1} \equiv \tilde{Z}_k(\rho_0)$ with ρ_0 minimum of $\tilde{U}_k$ and the subscript k has been dropped on dimensionless functions. We further introduce a dimensionless cutoff function $r(y) = Q_k(q^2)/(\bar{Z}_k q^2)$ where $y = q^2/k^2$ and hence $\partial_s Q_k = \bar{Z}_k k^2 s(y)$ with $s(y) = -\eta_Z y r(y) - 2y^2\partial_y r(y)$. Finally, the flow equations for the dimensionless functions u' and z are given by

$$\partial_s u' = u'(-2+\eta_Z) + (-2+d+\eta_Z)\bar{\rho}u'' + \frac{1}{2}(3u'' + 2\bar{\rho}u''')L_1^d + \frac{z'}{2}L_1^{2+d} \tag{23a}$$

$$\begin{aligned} \partial_s z = {} & z\eta_Z + (-2+d+\eta_Z)\bar{\rho}z' + \frac{1}{2}(z' + 2\bar{\rho}z'')L_1^d - 2\bar{\rho}z'(3u'' + 2\bar{\rho}u''')L_2^d \\ & + \frac{1}{d}\Big\{-(1+2d)\bar{\rho}z'^2 L_2^{2+d} + 2\bar{\rho}(3u'' + 2\bar{\rho}u''')^2 M_4^d \\ & + 4\bar{\rho}z'(3u'' + 2\bar{\rho}u''')M_4^{2+d} + 2\bar{\rho}z'^2 M_4^{4+d}\Big\} \end{aligned} \tag{23b}$$

$$\partial_s \ln X_k = \frac{1}{2}\bar{\rho}(3u'' + 2\bar{\rho}u''')^2 L_3^d + \bar{\rho}z'(3u'' + 2\bar{\rho}u''')L_3^{2+d} + \frac{1}{2}\bar{\rho}z'^2 L_3^{4+d} \tag{23c}$$

where primes denote derivatives wrt $\bar{\rho}$ and the so-called thresholds functions L and M are defined by

$$\begin{aligned} L_n^d &= -(n+\delta_{n0})v_d \int \mathrm{d}y\; y^{d/2-1}\frac{s(y)}{h(y)^{n+1}} \\ M_n^d &= v_d \int \mathrm{d}y\; y^{d/2}\left(\frac{-(n+\delta_{n0})s(y)(\partial_y h(y))^2}{h(y)^{n+1}} + \frac{2\partial_y s(y)\partial_y h(y)}{h(y)^n}\right) \end{aligned} \tag{24}$$

with $v_d^{-1} = 2^d\pi^{d/2}\Gamma(d/2)$ and $h(y) = y(z + r(y)) + u' + 2\bar{\rho}u''$. By definition (see section 4), the anomalous dimension $\eta_X \equiv -\partial_s \ln \bar{X}_k$ is obtained by evaluating equation (23*c*) at the running minimum ρ_0 of the potential $\tilde{U}_k$, or equivalently at the minimum $\bar{\rho}_0$ of u. Similarly, η_Z is obtained by solving the equation $\eta_z = -\rho_s \ln \bar{Z}_k = -(\rho_s z|_{\bar{\rho}_0} - z(\bar{\rho}_0)\,\eta_z + z'(\bar{\rho}_0)\,\rho_s\bar{\rho}_0)$ where $\rho_s z$ is given by (23*b*) and $z(\bar{\rho}_0) = 1$ by definition of $\bar{Z}_k$. The additional contribution $z'\partial_s\bar{\rho}_0$ is generated by the running of the minimum implicitly defined as $u'(\bar{\rho}_0) = 0$. Indeed, the running of u implies that its minimum flows according to $\partial_s u'(\bar{\rho}_0) = 0 = \partial_s u'|_{\bar{\rho}_0} + \partial_s\bar{\rho}_0 u''|_{\bar{\rho}_0}$ which, using (23*a*) evaluated at $\bar{\rho}_0$, yields the expression for $\partial_s\bar{\rho}_0$.

We emphasize that, as is to be expected from FDT, the dynamics decouples from the statics, i.e. the non-perturbative flow equations (23) for u' and z do not depend on X_k or η_X (and they identify with those derived in equilibrium with the Ansatz (13) [23]). Note that the

Table 1. Critical exponents of Model A in $d = 3$ from the different NPRG approximations (LPA and UZA) computed in this work, compared with results from other field theoretical methods (FT) and Monte Carlo simulations (MC).

$d = 3$	ν	η	z	Reference
LPA	0.65	0.11	2.05	This work
UZA	0.63	0.05	2.14	This work
FT	0.6304(13)	0.0335(25)		[26]
MC	0.6297(5)	0.0362(8)		[31]
FT			2.0237(55)	[32]
MC			2.032(4) 2.055(10)	[33, 34]

threshold functions L and M intervening in the equation for the anomalous dimensions can be computed analytically upon the choice of the cutoff $r(y) = (1/y-1)\theta(1-y)$ (corresponding to (17)), which greatly simplifies the numerical resolution. The numerical procedure to integrate the flow equations (23) is detailed in the next section.

6. Numerical integration of the flow equations

We will consider different levels of approximation. In a first step—which will be referred to as local potential approximation (LPA)—the field dependence of the kinetic renormalization function z can be neglected, i.e. only a running coefficient $\bar{Z}_k$ is considered. When restoring the $\bar{\rho}$-dependence of z, the corresponding approximation will be denoted by UZA.

For both approximations, the numerical procedure to determine the fixed point solution of equation (23) and to compute the critical exponents is the following. We sample the field $\bar{\rho}$ on a mesh of spacing $\Delta\bar{\rho}$ and discretize the flow equations (23) using finite differences at order $\Delta\bar{\rho}^4$ to calculate the $\bar{\rho}$-derivatives of u (and z). For the integrals, we either use their analytical expression (whenever available) or calculate them numerically using Simpson's rule. We implement an explicit forward integration scheme to propagate the solution between scale s and $s + \Delta s$, which turns out to be stable for sufficiently small $\Delta\bar{\rho}$ and Δs. The convergence of the numerical procedure when varying $\Delta\bar{\rho}$ and Δs has been carefully checked. We start out at the microscopic scale $s = 0$ ($k = \Lambda$) from a quartic bare potential $u(\bar{\rho}) = \lambda/2(\bar{\rho}-\bar{\rho}_\Lambda)^2$ where $\bar{\rho}_\Lambda$ is inversely proportional to the temperature. We carry through the numerical integration by lowering s towards $s \to -\infty$ ($k \to 0$). For small bare $\bar{\rho}_\Lambda$, the system flows to the symmetric (high temperature) phase where the (dimensionful renormalized) minimum of the potential $\rho_0 = k^{d-2}\bar{Z}_k^{-1}\bar{\rho}_0$ vanishes, whereas for large $\bar{\rho}_\Lambda$, it flows to the broken (low temperature) phase where ρ_0 acquires a finite value as $s \to -\infty$. For a fine-tuned initial $\bar{\rho}_\Lambda^c$, the system is in the critical regime, which corresponds to the effective potential u (and z) flowing to a fixed point (scale invariant) form u^* (and z^*). The critical exponents η and z can then be computed from the fixed point values of η_Z and η_X. The critical index ν is obtained by linearizing the flow in the vicinity of (u^*, z^*) and determining the (negative) eigenvalue characterizing the unstable (relevant) direction. This procedure is carried out within the LPA and UZA approximations. The results are gathered in tables 1 and 2 and are commented in the next section.

7. Results

The critical exponents for Model A obtained in this work from the NPRG equations (23) are summarized in tables 1 and 2 for dimensions $d = 3$ and $d = 2$ respectively,

Table 2. Critical exponents of Model A in $d = 2$ from the different NPRG approximations (LPA and UZA) computed in this work, compared with results from other field theoretical methods (FT) and Monte Carlo simulations (MC).

$d = 2$	ν	η	z	
LPA	0.78	0.43	2.15	This work
UZA	1.1	0.37	2.17	This work
Exact	1	1/4		
FT			2.0842(39)	[32]
MC			2.1667(5)	[35]

and compared with results ensuing from other field theoretical methods and Monte Carlo simulations.

Let us first comment on the equilibrium exponents η and ν. As emphasized in section 5, the dynamics decouples from the statics in equations (23)—as expected from TRS in the stationary regime. As a consequence, the equilibrium exponents computed here should match (up to numerical accuracy) those obtained in earlier NPRG works on the (equilibrium) Ising model. The three-dimensional Ising model has been thoroughly investigated within the NPRG framework as a testing ground of the method [28]. In particular, critical exponents have been computed using the cutoff function (17) in [25, 29] (though with different numerical procedures). The exponents ν and η we obtain in $d = 3$ precisely reproduce these values both at LPA and UZA.

We know from these previous studies that the accuracy can be improved by optimizing the choice of the cutoff function. At order ∇^2 in derivatives, optimized exponents are $\nu = 0.6281$ and $\eta = 0.0443$ [30], which are already in close agreement with the 6-loop calculations [26] or Monte Carlo simulations [31]. However, since the determination of η is related to the momemtum structure of the two-point correlation function, its accuracy is poorer than that of ν. A better accuracy on η requires to compute the next order ∇^4 in derivatives which yields $\eta = 0.033$ [25]. In two dimensions, far fewer NPRG results are available. A calculation with cutoff function (17) has been achieved in [29] and both results are in close agreement. As for in $d = 3$, the determination of η remains poorer than that for ν at order ∇^2 in derivatives. However, the two-dimensional Ising model has not been systematically investigated and neither optimized nor order ∇^4 exponents have been determined in $d = 2$.

We can now come to the new part of this work which concerns the dynamics. The situation for z is very different from that of the equilibrium critical exponents. For the dynamics, no high-loop expansions or exact results in $d = 2$ are available. Furthermore, results from MC simulations appear to be rather scattered especially in $d = 2$. The values reported in tables 1 and 2 seem to be accepted as reference values [16]. On the field-theoretical side, the determination of z is very sensitive to the choice of the resummation scheme since only a few orders in perturbation theory are known. Various resummation schemes have been studied leading to varied results and we quote here the latest results in [32].

Our results are in reasonable agreement with these various estimates. This is one of the key points of the NPRG approach and a central motivation for this work: the leading order in derivatives appears to already provide a reliable determination of physical quantities, as outlined above. Note that the variation on z between LPA and UZA is not meaningful and merely provides an estimate of the error. The reason is that going from LPA to UZA does not amount to enriching the Ansatz for the dynamical part since in both cases only a running coefficient X_k is allowed by TRS. Hence even at UZA, the accuracy on z remains poorer than that for the equilibrium exponents and the rapidity of convergence on z cannot be tested

within these approximations. One would need to implement the next order in time derivatives to improve the accuracy on z, which is rather costly. Alternatively, one could modify the regularization scheme and resort to a frequency and noise coarse-graining as mentioned in section 4, which is likely to yield better results, but is yet to be investigated.

8. Summary

In this work, we have studied the critical dynamics of Model A within the NPRG formalism. We have in particular detailed the construction of an appropriate approximation scheme preserving the symmetries, the derivation of the NPRG flow equations and their resolution. Using a very simple Ansatz, that is at the leading order in derivatives, we have obtained a reliable estimate for the dynamical exponent z: $z = 2.09(4)$ in $d = 3$ and $z = 2.16(1)$ in $d = 2$. The fact that the leading order already yields quantitative results is a generic feature of the NPRG approach [23] which makes it particularly powerful. This feature is the central motivation for the emphasis put throughout this work on the methodological part: a leading order NPRG calculation can already enable one to investigate nontrivial problems. Restricting to non-equilibrium critical phenomena, NPRG studies have indeed brought out new non-perturbative properties of reaction–diffusion processes [12] and allowed us to tackle interface growth problems [36]. We hence believe the approach to be useful to investigate many other non-equilibrium scaling phenomena. Of course, as the application of NPRG techniques to non-equilibrium statistical physics and dynamics is very recent, a great deal of systematic studies remain to be done as well to test the efficiency of the different Ansatz and regularization schemes out-of-equilibrium, which will be the goal of future works.

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Appendix. NPRG flow equation for Γ_k

This appendix is dedicated to the derivation of the exact flow equation (11) for Γ_k. (A similar derivation can be found for equilibrium systems in [23].) In this appendix, we use the shorthand $\mathbf{x} \equiv (x, t)$, vectors are denoted by capital letters (e.g. $\Psi \equiv [\psi, \tilde{\psi}]$, $\mathcal{J} = [j, \tilde{j}]$) and for functional derivatives we introduce the notation $\hat{\delta}_{\mathcal{J}_\mathbf{x}} \equiv [\delta/\delta j(\mathbf{x}), \delta/\delta \tilde{j}(\mathbf{x})]$ (and similarly for $\hat{\delta}_{\Psi_\mathbf{x}}$). Overhead hat symbols are used for (2×2) matrices.

The variation of Γ_k (at fixed Ψ) under an infinitesimal change of the running scale k follows from equation (10):

$$\partial_k \Gamma_k[\Psi_k]|_\Psi = -\partial_k \mathcal{W}_k[\mathcal{J}]|_\Psi + \int_\mathbf{x} \partial_k \mathcal{J}(\mathbf{x}) \cdot {}^t\Psi_k(\mathbf{x}) - \frac{1}{2} \int_{\mathbf{x},\mathbf{x}'} \Psi(\mathbf{x}) \cdot \partial_k \hat{R}_k(\mathbf{x}, \mathbf{x}') \cdot {}^t\Psi(\mathbf{x}') \qquad (A.1)$$

where $\mathcal{W}_k \equiv \log \mathcal{Z}_k$. The variation with k of $\mathcal{W}_k$ at fixed Ψ is related to that of $\mathcal{W}_k$ at fixed $\mathcal{J}$ by

$$\partial_k \mathcal{W}_k|_\Psi = \partial_k \mathcal{W}_k|_\mathcal{J} + \int_\mathbf{y} \partial_k \mathcal{J}(\mathbf{y}) \cdot {}^t\hat{\delta}_{\mathcal{J}_\mathbf{y}} \mathcal{W}_k. \qquad (A.2)$$

The expression of $\partial_k \mathcal{W}_k|_{\mathcal{J}}$ is obtained by taking the derivative of (9) wrt k:

$$\begin{aligned}\partial_k \ln \mathcal{Z}_k|_{\mathcal{J}} &= \int \mathcal{D}\Phi \left\{ -\frac{1}{2} \int_{\mathbf{x},\mathbf{x}'} \Phi(\mathbf{x}) \cdot \partial_k \hat{R}_k(\mathbf{x},\mathbf{x}') \cdot {}^t\Phi(\mathbf{x}') \right\} e^{-\mathcal{S}_k - \Delta\mathcal{S}_k + {}^t\Phi\mathcal{J}} \\ &= \left\{ -\frac{1}{2} \int_{\mathbf{x},\mathbf{x}'} \hat{\delta}_{\mathcal{J}_\mathbf{x}} \cdot \partial_k \hat{R}_k(\mathbf{x},\mathbf{x}') \cdot {}^t\hat{\delta}_{\mathcal{J}_{\mathbf{x}'}} \right\} e^{\mathcal{W}_k} \\ &= \partial_k \mathcal{W}_k e^{\mathcal{W}_k}.\end{aligned}$$

After expressing the derivatives $\hat{\delta}_{\mathcal{J}}$ and ${}^t\hat{\delta}_{\mathcal{J}}$ of $\exp(\mathcal{W}_k)$ and dividing out by $\exp(\mathcal{W}_k)$ one obtains

$$\partial_k \mathcal{W}_k|_{\mathcal{J}} = -\frac{1}{2} \int_{\mathbf{x},\mathbf{x}'} \left\{ \hat{\delta}_{\mathcal{J}_\mathbf{x}} \mathcal{W}_k \cdot \partial_k \hat{R}_k(\mathbf{x},\mathbf{x}') \cdot {}^t\hat{\delta}_{\mathcal{J}_{\mathbf{x}'}} \mathcal{W}_k + \mathrm{Tr}\left[\partial_k \hat{R}_k(\mathbf{x},\mathbf{x}') \cdot \hat{\delta}_{\mathcal{J}_\mathbf{x}} \left({}^t\hat{\delta}_{\mathcal{J}_{\mathbf{x}'}} \mathcal{W}_k \right) \right] \right\}. \quad \text{(A.3)}$$

The last term on the right-hand side is the matrix of second (functional) derivatives of $\mathcal{W}_k$ which we denote by $\hat{\mathcal{W}}_k^{(2)}(\mathbf{x},\mathbf{x}')$. The flow equation of Γ_k follows from inserting (A.2) and (A.3) in (A.1), which yields

$$\partial_k \Gamma_k[\Psi]|_\Psi = \frac{1}{2} \mathrm{Tr} \int_{\mathbf{x},\mathbf{x}'} \partial_k \hat{R}_k(\mathbf{x},\mathbf{x}') \cdot \hat{\mathcal{W}}_k^{(2)}(\mathbf{x},\mathbf{x}'). \quad \text{(A.4)}$$

This equation can be conveniently expressed in a closed form upon inverting $\hat{\mathcal{W}}_k^{(2)}$. The inverse of $\hat{\mathcal{W}}_k^{(2)}$ can be obtained by taking a functional derivative ${}^t\hat{\delta}_{\Psi_{\mathbf{x}'}}$ of the definition of the conjugate fields $\Psi(\mathbf{x}) = \hat{\delta}_{\mathcal{J}_\mathbf{x}} \mathcal{W}_k$:

$$\widehat{1} \cdot \delta^{(d+1)}(\mathbf{x},\mathbf{x}') = \int_{\mathbf{y}} {}^t\hat{\delta}_{\Psi_{\mathbf{x}'}} \mathcal{J}(\mathbf{y}) \cdot \hat{\mathcal{W}}_k^{(2)}(\mathbf{y},\mathbf{x}). \quad \text{(A.5)}$$

The matrix ${}^t\hat{\delta}_{\Psi_{\mathbf{x}'}} \mathcal{J}(\mathbf{y})$ is simply given by two successive derivatives $\hat{\delta}_{\Psi_\mathbf{y}}$ and ${}^t\hat{\delta}_{\Psi_{\mathbf{x}'}}$ of equation (10), which yields

$${}^t\hat{\delta}_{\Psi_{\mathbf{x}'}} \mathcal{J}(\mathbf{y}) = \hat{\Gamma}_k^{(2)}(\mathbf{x}',\mathbf{y}) + \hat{R}_k(\mathbf{x}',\mathbf{y}), \quad \text{(A.6)}$$

where $\hat{\Gamma}_k^{(2)}(\mathbf{x},\mathbf{x}')$ denotes the matrix of second (functional) derivatives of Γ_k. Finally, inserting (A.6) in the flow equation (A.4) yields the advocated equation:

$$\partial_k \Gamma_k[\Psi_k]|_\Psi = \frac{1}{2} \mathrm{Tr} \int_{\mathbf{x},\mathbf{x}'} \partial_k \hat{R}_k(\mathbf{x},\mathbf{x}') \cdot \left[\hat{\Gamma}_k^{(2)} + \hat{R}_k \right]^{-1}(\mathbf{x},\mathbf{x}'), \quad \text{(A.7)}$$

which can be Fourier transformed and underlies the NPRG calculations of this work.

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